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2004 J. Phys. A: Math. Gen. 37 10479

(http://iopscience.iop.org/0305-4470/37/44/002)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 10479–10495 PII: S0305-4470(04)81925-5

Ageing phenomena without detailed balance: the contact process

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Received 4 June 2004
Published 20 October 2004
Online at stacks.iop.org/JPhysA/37/10479
doi:10.1088/0305-4470/37/44/002

Abstract

The long-time dynamics of the 1D contact process suddenly brought out of an uncorrelated initial state is studied through a light-cone transfer-matrix renormalization group approach. At criticality, the system undergoes ageing which is characterized through the dynamical scaling of the two-times autocorrelation and autoresponse functions. The observed non-equality of the ageing exponents a and b excludes the possibility of a finite fluctuation–dissipation ratio in the ageing regime. The scaling form of the critical autoresponse function is in agreement with the prediction of local scale invariance.

PACS numbers: 05.70.Ln, 64.60.Ht, 75.40.Gb, 02.60.Dc

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Consider a statistical system prepared in some initial state. How does it relax towards one of its stationary states? This problem was first studied systematically in glassy systems where it was observed that the approach towards the thermodynamic equilibrium in this kind of system can be very slow (formally the relaxation time becomes infinitely large) and depends on the details of the history of how the relaxing material was treated. While this seemed to preclude any systematic study of such systems, it was found empirically [1] that the time dependence of observables can be cast into dynamical scaling forms such that universal and reproducible properties of the relaxation process emerge. Later, similar effects were also observed to occur in spin and structural glasses (see e.g. [2]) and in non-glassy, e.g.

simple ferromagnetic, systems[4]. From a microscopic point of view, the relaxation in these systems proceeds through the formation of correlated domains of a time-dependent typical size $\ell(t)$. This slow motion arises because any local discrete spin variable will have at least two distinct equilibrium values it could relax into. For a given position $\boldsymbol{r}$, the locally fluctuating variables (e.g. a local magnetic field $h(\boldsymbol{r})$ in magnetic systems) will select towards which of the possible equilibrium states the local order parameter $\phi(\boldsymbol{r})$ will evolve but there does remain a competition between the distinct equilibrium states at the macroscopic level. For simple ferromagnets and other non-glassy systems, it is generally admitted that $\ell(t) \sim t^{1/z}$, where z is the dynamical exponent. For systems relaxing towards an *equilibrium* steady state and which are brought infinitely rapidly from their initial state into contact with a thermal bath of temperature T, the value of z depends on whether $T < T_c$ or $T = T_c$, where T_c is the equilibrium critical temperature of the system. It is convenient to study these relaxation phenomena through two-time quantities such as the two-time autocorrelator $C(t, s)$ and the two-time linear autoresponse function $R(t, s)$ defined by

$$C(t, s) = \langle \phi(t)\phi(s) \rangle , \qquad R(t, s) = \left. \frac{\delta \langle \phi(t) \rangle}{\delta h(s)} \right|_{h=0} \tag{1}$$

where $\phi(t)$ is the time-dependent order parameter and $h(s)$ is the magnetic field conjugate to ϕ. Causality implies that $R(t, s) = 0$ for $t < s$. By definition, a system is said to undergo *ageing*, if either $C(t, s)$ or $R(t, s)$ does not merely depend on the time difference $\tau = t - s$, but on both the *observation time* t and the *waiting time* s. Ageing occurs for quenches to temperatures either below or at the critical temperature but systems in the high-temperature phase with $T > T_c$ do not age. For recent reviews, see e.g. [2–7].

Ageing systems may display dynamical scaling in the long-time limit [1–7]. Specifically, consider the two-time functions in the ageing regime $t \gg t_{\rm micro}$, $s \gg t_{\rm micro}$ and $\tau = t - s \gg t_{\rm micro}$, where $t_{\rm micro}$ is some microscopic time. Then one expects the scaling behaviour

$$C(t, s) \sim s^{-b} f_C(t/s), \qquad R(t, s) \sim s^{-1-a} f_R(t/s) \tag{2}$$

where the scaling functions $f_{C,R}(x)$ have the following asymptotic behaviour for $x \to \infty$:

$$f_C(x) \sim x^{-\lambda_C/z}, \qquad f_R(x) \sim x^{-\lambda_R/z}. \tag{3}$$

Here λ_C and λ_R are called the autocorrelation [8, 9] and autoresponse [10] exponents, respectively. The values of the exponents $\lambda_{C,R}$ and z depend on whether $T < T_c$ or $T = T_c$. For example, $z = 2$ for $T < T_c$ and a non-conserved order parameter. In general, the exponents λ_C and λ_R are distinct, but for an infinite-temperature initial state it can be shown that Galilei invariance of the model at zero temperature is a sufficient and model-independent criterion for the equality $\lambda_C = \lambda_R$ [11]. Indeed, for a disordered initial state, $\lambda_C = \lambda_R$ had been taken for granted since a long time, see e.g. [2, 3, 5], and for recent reconfirmations in interacting field theory see [12, 13]. In that case, one has $\lambda_C \geqslant d/2$ [14]. The exponents $\lambda_{C,R}$ are independent of the equilibrium exponents and of z [15].

For ageing ferromagnetic systems with a non-conserved order parameter, the value of the exponent a depends on the properties of the equilibrium system as follows [16, 17]. A system is said to be in *class S* if its order-parameter correlator $C_{\rm eq}(\boldsymbol{r}) \sim \exp(-|\boldsymbol{r}|/\xi)$ with a finite ξ and it is said to be in *class L*[5] if $C_{\rm eq}(\boldsymbol{r}) \sim |\boldsymbol{r}|^{-(d-2+\eta)}$, where η is a standard equilibrium critical exponent. Then

$$a = \begin{cases} 1/z; & \text{for class } S \\ (d - 2 + \eta)/z; & \text{for class } L. \end{cases} \tag{4}$$

[4] In these studies, it is always assumed that the underlying dynamics satisfies detailed balance.

[5] For example, the kinetic spherical model quenched to $T < T_c$ is in class L.

Furthermore, $b = 0$ for $T < T_c$ and $b = a$ if $T = T_c$, see e.g. [5–7]. Systems quenched to $T = T_c$ are always in class L.

The distance from equilibrium is conveniently measured through the *fluctuation–dissipation ratio* [18, 19]

$$X(t,s) := TR(t,s)\left(\frac{\partial C(t,s)}{\partial s}\right)^{-1}. \tag{5}$$

At equilibrium, the fluctuation–dissipation theorem states that $X(t,s) = 1$. Ageing systems may also be characterized through the limit fluctuation–dissipation ratio [5, 20, 21]

$$X_\infty = \lim_{s\to\infty}(\lim_{t\to\infty} X(t,s)). \tag{6}$$

Below criticality, one expects $X_\infty = 0$, but if $T = T_c$, it should be a universal number, and this has been confirmed in a large variety of systems in one and two space dimensions [12, 21–26]. Alternatively, at $T = T_c$ one may fix $x = t/s$ and consider $X(x) = \lim_{s\to\infty} X(xs,s)$ and then $X_\infty = \lim_{x\to\infty} X(x)$. The order of the limits is important, since $\lim_{t\to\infty}(\lim_{s\to\infty} X(t,s)) = 1$.

The above discussion of the properties of ageing systems has implicitly assumed that the system's dynamics satisfies detailed balance and therefore always relaxes towards an *equilibrium* steady state even if it may never reach it. Here we wish to investigate the relaxation of more general systems where detailed balance is no longer satisfied and whose steady state therefore cannot be in thermodynamic equilibrium. Probably the simplest kinetic system fundamentally far from equilibrium is the celebrated *contact process* which has a steady-state transition in the directed percolation universality class[6].

In section 2, we shall recall the definition of the model and discuss the computation of correlators and response functions before we present in section 3 those elements of the light-cone transfer-matrix renormalization group (LCTMRG) which are important for our purposes. Calculating two-time observables from the LCTMRG, we study in section 4 their time-dependent scaling behaviour. In particular, we shall investigate whether there exist scaling forms analogously to equations (2,3) and if so, what the values of the ageing exponents $a, b, \lambda_C/z, \lambda_R/z$ are. We shall inquire whether any analogue of a finite fluctuation–dissipation ratio might exist. This question is of particular importance since in a two-dimensional voter model an effective non-equilibrium temperature can indeed be defined through equation (5) [25]. However, the voter model still satisfies detailed balance (albeit in a slightly unusual form) and we shall try and see whether the procedure proposed in [25] might conceivably be extended to the contact process. A more quantitative question concerns the form of the time-dependent scaling functions. Indeed, for ageing ferromagnets it has been shown that dynamical scaling as defined above can be extended to a richer *local scale invariance* [28]. In particular, the scaling forms of the two-time response function $f_R(y)$ [29, 30] and more recently also of the two-time autocorrelation function $f_C(y)$ for $T < T_c$ [11, 31] have been derived. We shall study whether an analogous generalization is possible in the contact process. A complementary paper studies the same model through intensive Monte Carlo simulations [32]. In section 5, we conclude.

2. The contact process

The 1D contact process (CP) is defined as follows and might be conceived as a simple model describing the propagation of an epidemic disease, see [33, 34] for reviews. Consider a chain

[6] See [27] for an exactly solvable kinetic Ising model with a non-equilibrium steady state.

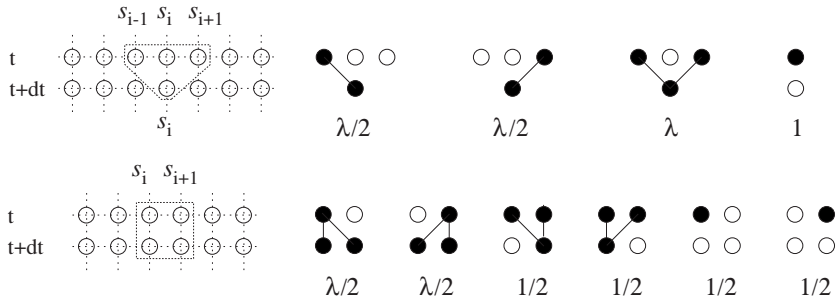

Figure 1. Rules for the contact process on a 1D lattice. Upper line: updating rules for a Monte Carlo simulation. Lower line: updating rules for the LCTMRG.

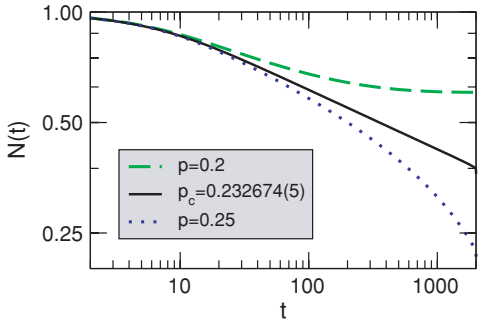

Figure 2. Density $N(t)$ of the 1D contact process in the absorbing phase ($p = 0.25$), at criticality ($p = p_c$) and in the ordered phase ($p = 0.2$). Time is measured in units of the increment $\Delta t = 0.02$.

where each site can either be empty ('healthy') or else be occupied by a particle of a single species A ('infected'). The evolution occurs according to the following rules and rates:

$$A\emptyset, \emptyset A \xrightarrow{\lambda} AA \text{ ('infection')} \tag{7}$$

$$A \xrightarrow{1} 0 \text{ ('healing')}. \tag{8}$$

We shall consider throughout the case of one spatial dimension. Then an empty site next to an occupied one is infected with rate λ, while infected sites heal at rate 1, independently of their neighbours. In the upper panel of figure 1, we recall the elementary processes for updating a 1D lattice with the corresponding rates and this readily defines a Monte Carlo algorithm with asynchronous random sequential updates. In the lower panel, we present a version of the CP which is more adapted for use with the LCTMRG, see section 3. Besides λ, it is also common to parametrize the contact process with $p := (1 + \lambda)^{-1}$ as we shall do in the following: if a lattice site i is occupied, the particle is annihilated with probability p, while if the site is empty, a particle is created at one of the neighbouring sites of i with probability $1 - p$.

There is a steady-state phase transition between the ordered phase ('infected') at small p (large λ) and the absorbing phase ('healthy') at large p (small λ). To illustrate this, consider the average density $N(t) := \langle n_i(t) \rangle$ which is in fact independent of the lattice site i and which is computed using the methods described in section 3. In figure 2, $N(t)$ is shown for three values of p. There is a critical value $p_c = 0.232\,674(5)$ ($\lambda = \lambda_c = 3.297\,85(8)$) [33] such that

$N(t) \sim t^{-\delta}$. The exponent $\delta \simeq 0.16$ can be read off from the slope in figure 5, in agreement with the literature [33]. For $p < p_c$ ($\lambda > \lambda_c$), the density converges exponentially fast towards a positive steady-state value $N_\infty := \lim_{t\to\infty} N(t) > 0$, while for $p > p_c$ ($\lambda < \lambda_c$), the density decays exponentially fast towards zero.

Correlations are defined as density–density autocorrelators, both in a disconnected and a connected form

$$C(t,s) := \langle n_i(t) n_i(s)\rangle, \qquad \Gamma(t,s) := C(t,s) - N(t)N(s). \tag{9}$$

In order to obtain the autoresponse function

$$R(t,s) = \left.\frac{\delta\langle n_i(t)\rangle}{\delta h_i(s)}\right|_{h_i=0} \tag{10}$$

we must introduce an external field h_i coupled to the density operator n_i at site i. The most straightforward possibility is to introduce a spontaneous creation of particles $\emptyset \longrightarrow A$ on site i with rate h_i.

Previous experience with $R(t,s)$ comes from magnets, where the system's response to an external field is studied via the integrated response since data for $R(t,s)$ are usually too noisy. For the integrated response, the most common protocols in magnets are zero-field-cooled (ZFC) and thermoremanent magnetization (TRM), but also intermediate protocols have been proposed. However, the analysis of the time-dependent scaling of integrated response functions is not always straightforward and the scaling behaviour of interest may even be obscured by non-scaling terms or finite-time corrections. This occurs in particular for quenches into the ordered phase [17]. Another difficulty is that the application of a uniform external field h over several time steps introduces extra particles into the lattice which brings the steady state into the active phase. For magnets, sophisticated techniques have been proposed [35–38] in order to avoid such a systematic change of the system's properties and which may involve random external fields which change sign in every time step, followed by an average over realizations of the randomness or delicate changes in the dynamics. Due to the absence of detailed balance in the contact process, these methods are not available here.

In view of these difficulties, it is a great advantage of the LCTMRG method that the autoresponse can be computed directly (see section 3). This is faster, more accurate and conceptually simpler. The limit $h \to 0$ can be taken analytically so the problems created by a non-vanishing external field are circumvented.

In calculating $C(t,s)$, $\Gamma(t,s)$ and $R(t,s)$, we shall always use a completely filled lattice as initial state. This state is completely uncorrelated and can be thought of as prepared at $p = 0$ ($\lambda = \infty$).

Since the CP does not contain *a priori* a temperature variable, we might try and define an analogy of the fluctuation–dissipation ratio as follows:

$$X(t,s) := \frac{R(t,s)}{\partial\Gamma(t,s)/\partial s}. \tag{11}$$

3. The LCTMRG method

Instead of the usual Monte Carlo (MC) simulation of the time evolution of the contact process, see [32], we use a new variant of the density-matrix renormalization group (DMRG) applied to stochastic transfer matrices, the so-called light-cone transfer-matrix density-matrix renormalization group (LCTMRG) [39]. This algorithm, which is an improvement on earlier DMRG approaches to calculate the time evolution of stochastic systems [40, 41] has the following advantages:

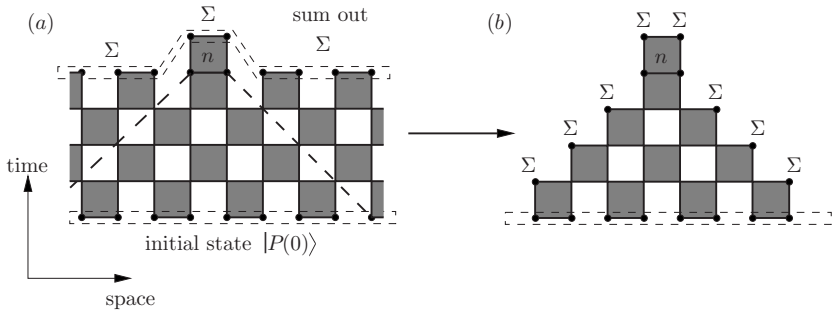

Figure 3. (*a*) Trotter–Suzuki decomposition of $2\Delta t$ time steps. The resulting 2D lattice consists of local plaquette interactions τ and is infinitely extended in space direction. The dimension of the time direction is finite and the boundary conditions are fixed by $\langle 1|$ and $|P(0)\rangle$. (*b*) Reduction of the 2D lattice to a triangle structure. All other plaquettes trivialize, i.e. do not contribute to the state of the top of the triangle. After [39].

- There is no need for random numbers and ensemble averages since *all* relevant ensembles and correlations (in the sense explained below) are taken into account. One LCTMRG run takes a few minutes while 1000 MC runs may take days. The resulting correlation functions are very smooth and require no further statistics, e.g. in order to compute numerical derivatives with respect to time like $\partial\Gamma(t, s)/\partial s$.
- The transfer matrix enables us to take the thermodynamic limit $L \to \infty$ *exactly*.

However, the LCTMRG is still plagued by numerical instabilities whose exact origin is unclear, restricting the calculation to about 1000 time steps. The LCTMRG is not very useful for models where each site may have many different states ($n \gg 2$), or where the interaction spans more than two or three sites.

The dynamics of the one-dimensional stochastic process can be mapped by a Trotter–Suzuki checkerboard decomposition onto a two-dimensional classical model: this is the geometric interpretation of percolation in two spatial dimensions, directed along one of the two axes. The checkerboard is made up of plaquettes τ ('local transfer-matrices') encoding the local interaction according to the rules in figure 1:

$$(\tau)^{l_1 l_2}_{r_1 r_2} = \langle l_2 r_2|\, \mathrm{e}^{-\Delta t \cdot h} |l_1 r_1\rangle = \quad \text{with} \quad l_i, r_i \in \{0, 1\} \tag{12}$$

where h is the local transition-rate matrix from two neighbouring sites $l_1 r_1$ at time t to the same sites at time $t + \Delta t$. The time step $\Delta t \ll 1$ should be chosen sufficiently small.

We determine the thermodynamic properties of the system by a transfer matrix: this ensures that the system is truly infinite in space, while we can follow the short-time dynamics for a certain number of time steps.

Because of probability conservation (equation (13)) and causality (at each time step, only a neighbouring site may be affected by the local interaction), the measurement of a local observable $n_i(t)$ at time step t and site i depends only on the 'past light cone' of this site on the classical 2D lattice ([41]; see figure 3).

$$\forall l_1, r_1 : \sum_{l_2 r_2} (\tau)^{l_1 l_2}_{r_1 r_2} = 1, \qquad = 1. \tag{13}$$

As the dimension of the exact transfer matrix grows exponentially with the number of time steps, we use the DMRG idea to decimate the state space. The DMRG relies on splitting

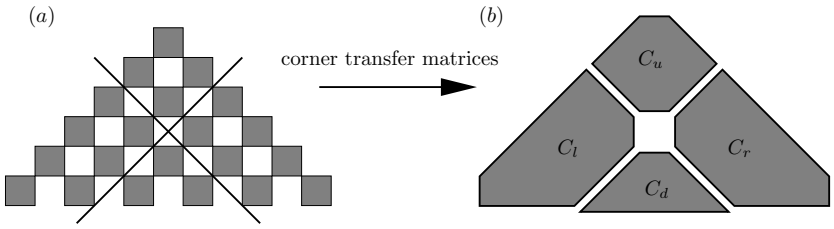

Figure 4. Splitting the light cone into four corner transfer matrices (after [39]).

the system into two strongly correlated parts, called the 'system' and 'environment'. Kemper *et al* [39] have proposed an efficient realization of the DMRG algorithm applied to corner transfer matrices. These are obtained by diagonal cuts through the checkerboard: the light cone is split into four parts diagonally along the future and past light cone of the centre point of the triangle (see figure 4). For details we refer to [39].

We made a modification to the algorithm in order to compute the autocorrelation and autoresponse functions necessary for investigating the ageing behaviour. Usually for a local observable $n_i(t)$ the expectation value is obtained by multiplying the local transfer matrix at site i and time t with $n_i(t)$ before applying the initial and final conditions and taking the trace over temporal indices. For the two-time correlation function $C(t, s)$, the algorithm has been modified to multiply the local transfer matrices τ adjacent to site i with n_i both at time steps s and t before the trace. From $C(t, s)$ the connected autocorrelation $\Gamma(t, s)$ is computed via (9), and the derivative of the connected autocorrelation function is computed from a symmetric difference, i.e.

$$\frac{\partial \Gamma(t,s)}{\partial s} := \frac{\Gamma(t, s + \Delta t/2) - \Gamma(t, s - \Delta t/2)}{\Delta t} \tag{14}$$

which is sufficiently accurate (i.e. independent of Δt for $\Delta t = 0.01 \ldots 0.05$).

Likewise, when applying an external field h_i in order to compute $R(t, s)$, the local τ adjacent to site i at time step $t = s$ is modified to include particle production at rate h_i. However, as we are interested in the derivative with respect to the external field, it is better to compute this derivative analytically: the Hamiltonian in the presence of an external field h_i on site i is

$$H_{h_i} = H + \mathrm{id}_{\ldots,i-1} \otimes h_i \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}_i \otimes \mathrm{id}_{i+1,\ldots}$$

where H is the stochastic Hamiltonian for the CP, see e.g. [42, 43] for reviews. Then using the state at time $t = s$, $|P(s)\rangle = \mathrm{e}^{-Hs}|P(t = 0)\rangle$, and the final state $\langle 1|$,

$$\begin{aligned} R(t,s) &= \lim_{h_i, \Delta t' \to 0} \langle 1| \mathrm{e}^{-H(t-s-\Delta t')} \left(\frac{\mathrm{e}^{-H_{h_i}\Delta t'} - \mathrm{e}^{-H\Delta t'}}{h_i \Delta t'} \right) |P(s)\rangle \\ &= \lim_{h_i, \Delta t' \to 0} \langle 1| \mathrm{e}^{-H(t-s-\Delta t')} \left(\frac{(H - H_{h_i})\Delta t' + \mathcal{O}((\Delta t')^2)}{h_i \Delta t'} \right) |P(s)\rangle \\ &= \langle 1| \mathrm{e}^{-H(t-s)} \left(\mathrm{id}_{\ldots,i-1} \otimes - \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}_i \otimes \mathrm{id}_{i+1,\ldots} \right) |P(s)\rangle \end{aligned}$$

where the matrix is written in terms the local basis $(0, A)$ on site i. This has two advantages:

(i) The limit $h_i \to 0$ is taken exactly, thus there is no danger of triggering a phase transition by inserting extra particles into the system.

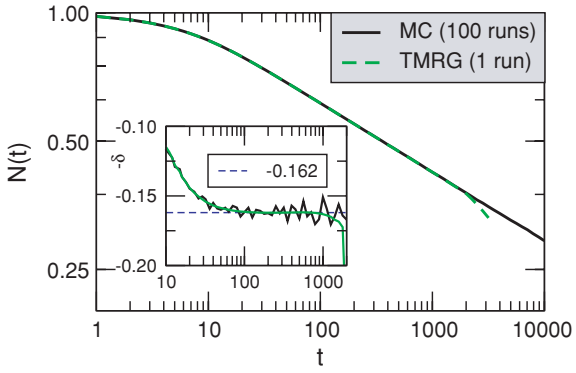

Figure 5. Comparison of the mean particle density $N(t)$ found by MC and by the LCTMRG for the critical 1D contact process, $p = p_c$. The slope $-\delta$ is shown in the inset. The LCTMRG becomes numerically unstable around $t = 1000$. Times are measured in units of $\Delta t = 0.02$.

(ii) No numerical derivative is necessary which would have included the difference of two very similar quantities, so this method is numerically more accurate.

In figure 5, we compare the results for $N(t)$ of the critical contact process in 1D obtained from MC or the LCTMRG, respectively. First, we observe that the LCTMRG data are fairly smooth, as is especially evident when studying the exponent δ directly. However, we also see that the LCTMRG becomes numerically unstable around $t = 1000$ time steps. This happens because the basis vectors of the reduced state space offered during the renormalization by the LCTMRG method step become inadequate: the expectation value of the identity operator $\langle 1 \rangle$ is around 1 (as it should be) only for the first several hundred time steps, then decreases to below 0.1. However, the onset of instability can in practise always be identified very reliably. The reason for this instability is that DMRG works best if system and environment are quite strongly entangled, which is not the case here.

4. Results

In this section, we first present the results of our numerical calculations. In subsections 4.4 and 4.5, we shall discuss two important implications of our numerical results.

The MC results in figure 5 are obtained using the 3-site update in the first row of figure 1 with asynchronous dynamics and parallel update, system size $L = 10^5, 10^4$ time steps and ensemble averaging over 100 runs. All LCTMRG results are obtained with the 2-site update shown in the second row of figure 1, using the following parameters: time step size $\Delta t = 0.02$, number of states in projected state space $m = 32$, external field strength exactly $h \to 0$, lattice size $L = \infty$ (exact thermodynamic limit).

4.1. Ageing at criticality

For the critical contact process, we now try and see whether the scaling forms used to describe ageing in magnets apply. We consider the scaling forms

$$C(t,s) = s^{-b} f_C(t/s), \qquad f_C(y) \sim y^{-\lambda_C/z} \tag{15a}$$

$$\Gamma(t,s) = s^{-b} f_\Gamma(t/s), \qquad f_\Gamma(y) \sim y^{-\lambda_\Gamma/z} \tag{15b}$$

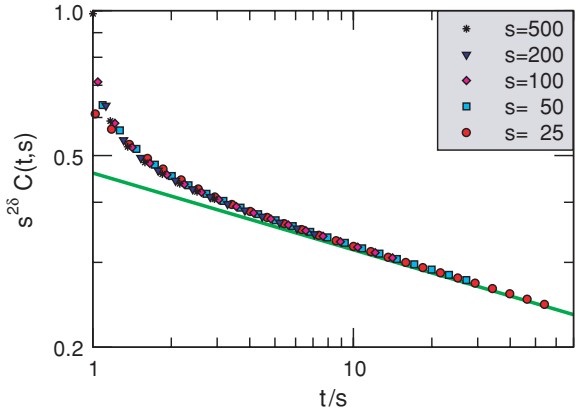

Figure 6. Autocorrelation $C(t, s)$ for the critical CP for several values of the waiting time s as a function of the scaling variable $y = t/s$. The straight line has the slope $-0.16 \approx -\delta$.

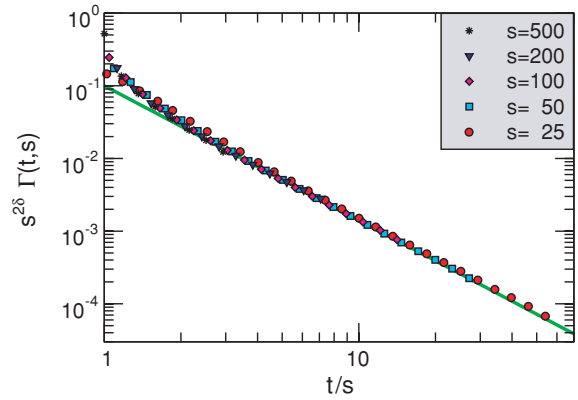

Figure 7. Connected autocorrelation $\Gamma(t, s)$ for the critical CP for several values of the waiting time s. The slope of the straight line is -1.85.

$$R(t, s) = s^{-1-a} f_R(t/s), \qquad f_R(y) \sim y^{-\lambda_R/z} \tag{15c}$$

where the asymptotic behaviour holds for $y \to \infty$. First, we display in figure 6 the critical autocorrelator. A nice data collapse is observed with an exponent $b = 2\delta$ and for large values of t/s, we find a power law according to equation (15) with $\lambda_C/z = \delta$.[7] This confirms the expectations derived in [32].

Of course, we are interested in the deviations from this as measured by the connected autocorrelator $\Gamma(t, s)$ which is shown in figure 7. Again, the data collapse very well and we obtain the ageing exponents $b = 2\delta$ and $\lambda_\Gamma/z = 1.85(10)$.

Next, we study the autoresponse function which is displayed in figure 8. As for the correlations, we find a neat dynamical scaling behaviour and for large values of $y = t/s$, we can read the exponents $1+a = 2\delta$ and $\lambda_R/z = 1.85(10)$. We shall come back in subsection 4.5 to a quantitative comparison of the precise functional form of the scaling function $f_R(y)$ with the prediction of local scale invariance.

[7] Here and in what follows, the numbers in brackets give the estimated error in the last given digit(s).

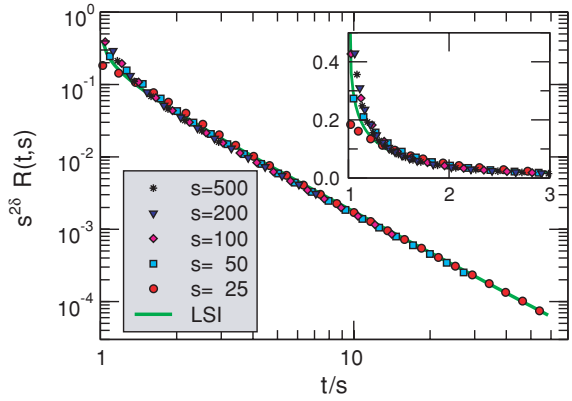

Figure 8. Autoresponse function $R(t, s)$ for the critical CP for several values of the waiting time s. The full curve labelled LSI is the prediction of local scale invariance $f_R(y) = 0.12 \cdot y^{-1.85}(1 - 1/y)^{-2\delta}$. The inset shows the same data for $1 \leqslant t/s \leqslant 3$.

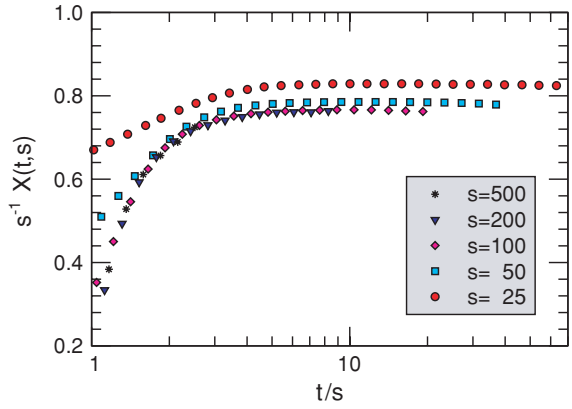

Figure 9. Fluctuation–dissipation ratio (FDR) $X(t, s)$ as defined in equation (11) for the critical contact process. For large waiting times $s \gtrsim 100$, the curves $s^{-1}X(t, s)$ collapse and quickly saturate at $\approx 0.77(1)$.

These tests provide evidence in favour of a dynamical scaling of the ageing behaviour in the contact process which at least formally is quite analogous to the one found previously in glasses and ferromagnets. However, there are qualitative differences between these models and the contact process as becomes explicit when considering the exponent relations

$$\lambda_\Gamma = \lambda_R, \qquad 1 + a = b = 2\delta. \tag{16}$$

In magnetic systems, the first of these had been simply taken for granted and only recently, it has been established (i) that it need not hold for spatially long-ranged initial correlations [10] or in certain glassy systems [44] and (ii) that for phase ordering (hence $z = 2$) with short-ranged initial correlations local scale invariance is sufficient to actually prove the exponent equality $\lambda_\Gamma = \lambda_R$ [11]. On the other hand, the *non-equality* of the ageing exponents a and b at a critical point comes as a surprise. This important result can be illustrated in a different way by considering the analogue equation (11) of the fluctuation–dissipation ratio. For the critical CP, we find from figure 9 for $s \to \infty$ a scaling behaviour $X(t, s) \simeq s \cdot f_X(t/s)$ where the scaling function $f_X(y)$ has a finite limit value $f_X(\infty) \simeq 0.77(1)$.

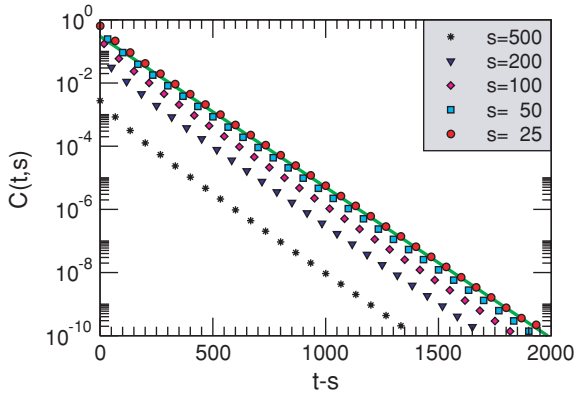

Figure 10. Autocorrelation $C(t, s)$ as a function of $\tau = t - s$ for several values of s in the absorbing phase at $p = 0.6$. The full curve is proportional to $\exp(-0.01(t - s))$.

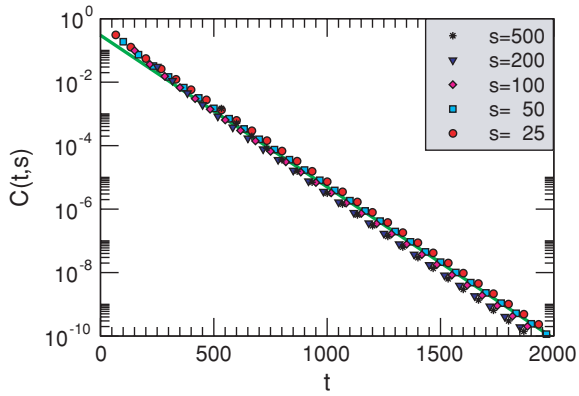

Figure 11. Autocorrelation $C(t, s)$ as a function of t for the CP with $p = 0.6$. The full curve is proportional to $\exp(-0.01t)$.

4.2. Absorbing phase

Here, we discuss the behaviour in the subcritical or absorbing phase, with $p = 0.6 > p_c$ ($\lambda < \lambda_c$). The asymptotic behaviour in this phase may be understood by considering the extreme case $p = 1$ first. For $p = 1$, the particles on different sites are uncorrelated and simply decay with a fixed rate. For any fixed site i and with $t > s$, a moment's thought shows that $n_i(t)n_i(s) = n_i(t)$ because $n_i \in \{0, 1\}$. Therefore, $C(t, s) = N(t)$.

This long-time behaviour of the autocorrelation survives in the entire absorbing phase, as is illustrated in figures 10 and 11 for $p = 0.6$. We see that $C(t, s)$ decays exponentially fast but there is no collapse as a function of $\tau = t - s$. (We point out that for intermediate values of s, the data appear to effectively collapse. For p close to p_c, the cross-over to the truly asymptotic behaviour may set in very late.) On the other hand, when plotted as a function of t, there is a collapse for large values of s, see figure 11. Similarly, $\Gamma(t, s) = C(t, s) - N(t)N(s) \sim N(t)$ for s sufficiently large.

On the other hand, for the response function we find in figure 12 a rapid collapse in terms of the time difference $\tau = t - s$ for larger waiting times $s \gtrsim 100$ and time-translation invariance is recovered, as it should be.

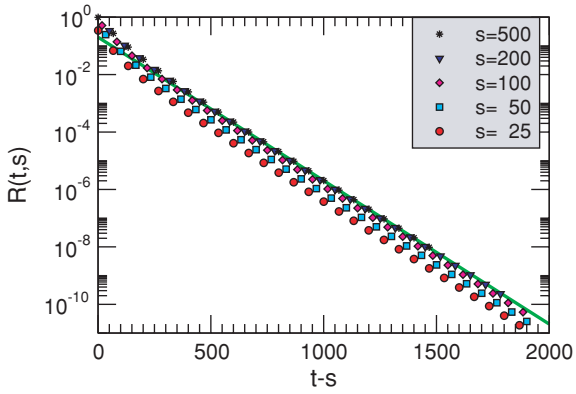

Figure 12. Autoresponse $R(t, s)$ as a function of $\tau = t - s$ for several values of s in the absorbing phase at $p = 0.6$. The full curve is proportional to $\exp(-0.01(t - s))$.

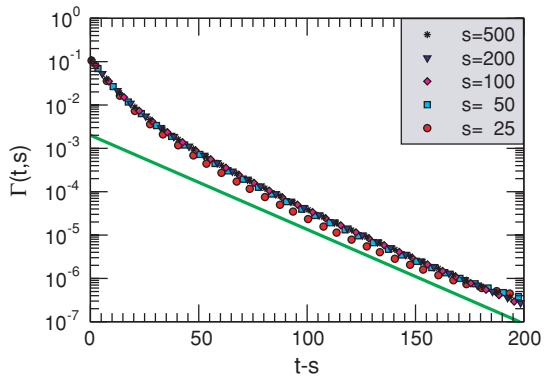

Figure 13. Connected autocorrelation $\Gamma(t, s)$ in the active phase of the contact process (with $p = 0.1$). The straight line is proportional to $\exp(-0.05(t - s))$.

4.3. Active phase

Finally, we discuss the long-time behaviour of the two-time observables in the active (percolating/ordered) phase, where $p = 0.1 < p_c$ ($\lambda = 9 > \lambda_c$). In distinction with magnetic systems, where ageing also occurs for quenches into the ordered phase, in the contact process time-translation invariance is rapidly recovered and no ageing occurs. We illustrate this in figure 13 for the connected autocorrelator and in figure 14 for the response function. In both cases, when plotted against $\tau = t - s$, we observe a collapse for waiting times $s \gtrsim 50$ and time-translation invariance is recovered.

4.4. Effective temperature

One of the main peculiarities of equilibrium states with respect to more general steady states is that their probability distribution has the form of a Boltzmann weight which may be characterized in terms of a temperature. It is therefore a natural question whether a *non-equilibrium* temperature might be defined for more general steady states. Indeed, such an attempt has been presented recently by Sastre *et al* [25]. They considered the fluctuation–

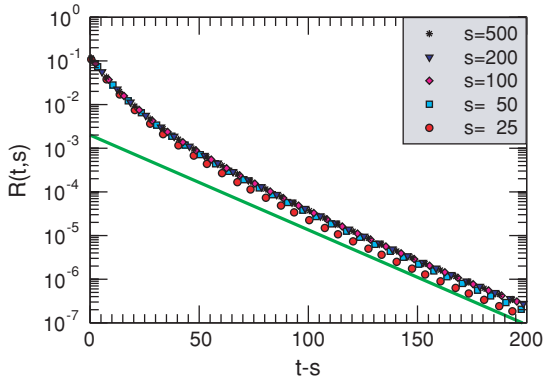

Figure 14. Autoresponse $R(t, s)$ in the active phase of the contact process (with $p = 0.1$). The straight line is proportional to $\exp(-0.05(t - s))$.

dissipation ratio equation (5) and observed that in the limit $s \to \infty$ and $t - s \to 0$, one should have an equilibrium-like regime, hence $X(t, s) \to 1$. From this observation, they define a *dynamical temperature* by

$$\frac{1}{T_{\rm dyn}} := \lim_{t\to\infty} \left(\lim_{t-s\to 0} \frac{R(t, s)}{\partial C(t, s)/\partial s} \right). \tag{17}$$

By explicit calculation, they confirm that in the 2D critical voter model this limit exists, has a non-trivial value and is universal [25]. However, the basic assumption of the idea of Sastre *et al* has been critically re-examined by Mayer and Sollich [45] who construct in the 1D Glauber–Ising model undergoing coarsening a defect-pair observable such that the fluctuation–dissipation ratio $X(t, s) \neq 1$ in the short time regime (in particular they show $\lim_{s\to\infty} X(s, s) = 3/4$).

We now ask whether definition (17) [25] could be extended to the contact process. Obviously, for the contact process the connected correlator $\Gamma(t, s)$ must be used in equation (17). Before we do so, it may be useful to recall some constraints on the ageing behaviour which hold for relaxation towards equilibrium. Equilibrium states are steady states (hence time-translation invariant) which satisfy in addition the fluctuation–dissipation theorem (thus $X(t, s) = 1$). This last condition may also be replaced by the Onsager symmetry condition $\langle A(t)B(s)\rangle_{\rm eq} = \langle B(t)A(s)\rangle_{\rm eq}$ for any two observables A and B; which together with time-translation invariance is enough to reproduce the fluctuation–dissipation theorem [19, 6].

We are interested in equilibrium critical dynamics. Combining the scaling forms of section 1 and time-translation invariance, we expect

$$C(t, s) \sim (t - s)^{-b}, \qquad R(t, s) \sim (t - s)^{-1-a}. \tag{18}$$

Combining this with the FDT, it follows $a = b$. Therefore, the equality $a = b$ is a necessary condition that the quasi-stationary state found when $t - s \ll 1$ is indeed a quasi-equilibrium one.

We can apply this argument to the quasi-steady state of the contact process characterized by the connected autocorrelator $\Gamma(t, s)$ (instead of $C(t, s)$) and the autoresponse $R(t, s)$. In subsection 4.1, we have seen that dynamical scaling equation (15) holds with the ageing exponents satisfying $1 + a = b = 2\delta$. Consequently, since a necessary condition for the existence of a quasi-equilibrium regime is *not* satisfied in the contact process, definition (17) of a non-equilibrium temperature cannot be extended to this universality class. It is remarkable to

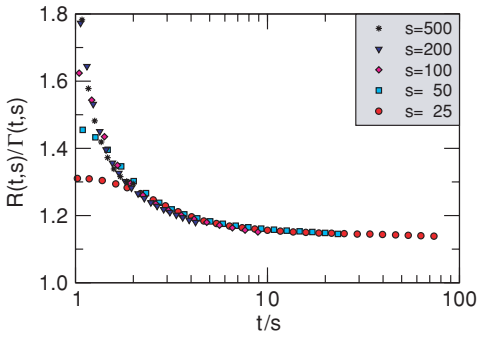

Figure 15. Ratio $R(t,s)/\Gamma(t,s)$ for the critical CP. For $s > 25$, the curves appear to collapse to a scaling function which in turn saturates at a finite value for $t/s \gtrsim 10$.

see that the presence or absence of detailed balance leads to very different response behaviour, once the dynamical scaling regime is reached. We conclude that definition (17) proposed in [25] is likely to reflect peculiar properties of the critical voter model rather than being generic.

Indeed, our data suggest that not the 'fluctuation–dissipation ratio' $X(t,s)$ as defined in equation (11), but rather the ratio

$$\Xi(t,s) := \frac{R(t,s)}{\Gamma(t,s)} \tag{19}$$

should become a scale-invariant function of t/s in the ageing regime at criticality, namely $\Xi = \Xi(t/s)$. This appears to be the case as is shown in figure 15 and for $t/s \to \infty$ we read off the limit value $\Xi_\infty \simeq 1.15(5)$. In analogy of the universal limit fluctuation–dissipation ratio X_∞ [21] in systems with detailed balance, it would be interesting to see whether or not Ξ_∞ is universal (for example by going beyond the mean-field treatment [32] of Reggeon field theory).

4.5. Local scale invariance

Having seen that at criticality, the ageing of the contact process satisfies dynamical scaling, it is natural to inquire whether the recently proposed extension [28] of dynamical scaling to a spacetime-dependent, i.e. *local* kind of scale invariance in magnets might also apply to the model at hand. The central assumption of that theory is that the linear response function formed from the so-called *quasi-primary* fields transforms covariantly under the action of the group of local scale transformations. We propose to try and see whether the form of the linear response in the critical contact process may be understood this way.

Indeed, for any given value of z, infinitesimal local scale transformations with a spacetime-dependent rescaling factor $1 + \varepsilon(t, \boldsymbol{r})$ can be constructed. In particular, the following explicit expression for the response function is obtained from the condition that $R(t,s)$ transforms covariantly under the action of local scale transformations [28, 29]

$$R(t,s) = r_0 \left(\frac{t}{s}\right)^{1+a-\lambda_R/z} (t-s)^{-1-a} \tag{20}$$

and where r_0 is a normalization constant. This prediction has been confirmed in several models with a dynamics given by a master equation, notably the kinetic Ising model with Glauber dynamics, both in the bulk [29, 30] as well as close to a free surface [46], the kinetic XY model with a non-conserved order parameter [47, 11] and for the Hilhorst–van Leeuwen model [46].

Table 1. Non-equilibrium exponents as defined in (15) for the ageing of the critical contact process in several space dimensions d, according to Monte Carlo [32] or LCTMRG (this work) calculations. The exponents following from mean-field theory [32] are also included.

d	a	b	λ_C/z	λ_Γ/z	λ_R/z	Method
1	−0.68(5)	0.32(5)	0.16(1)	1.85(10)	1.85(10)	LCTMRG
	−0.57(10)	0.319	0.159	1.9(1)	1.9(1)	Monte Carlo
2	0.3(1)	0.901(2)	0.450	2.8(3)	2.75(10)	Monte Carlo
>4	$\frac{d}{2}-1$	2	1	–	$\frac{d}{2}+2$	Mean field

Local scale invariance has also been confirmed for several variants of the exactly solvable spherical model and the free random walk [29, 21, 48, 11].

All these tests are based on a Master equation or on a linear Langevin equation which reduces to a free field theory. On the other hand, field-theoretical calculations of ageing ferromagnetic systems based on nonlinear Langevin equations find small corrections to equation (20) [12, 13]. However, it is not completely clear whether a description of ageing in terms of a Master equation and in terms of a Langevin equation are completely equivalent. A counterexample is provided by the 1D kinetic Ising model with Glauber dynamics at zero temperature where the universal exponent $\lambda_C = 1$ is known exactly [20, 49]. On the other hand, the Langevin equation usually believed to be equivalent to this model, namely the time-dependent Ginzburg–Landau equation, leads to the exact result $\lambda_C = 0.6006\ldots$ [50].

Comparing equation (20) with the LCTMRG data for the 1D critical contact process, we find from figure 8 a perfect agreement almost down to $t/s = 1$, where values for the exponents a and λ_R/z determined previously were used[8]. The contact process hence provides the first example of a model satisfying local scale invariance which has a non-equilibrium steady state.

5. Conclusions

We have studied the question if an analogue of the ageing phenomenon well known in magnetic systems also occurs in systems which relax to a non-equilibrium steady state. Our case study of the one-dimensional contact process has allowed us to answer this general question affirmatively, at least in situations when the steady state is critical and might be hence viewed as being formed from two coalescing steady states. On the other hand, in the active phase, where only a *single* stable steady state exists, time-translation invariance is rapidly recovered and no ageing occurs. At criticality, the ageing behaviour can be described in terms of dynamical scaling, see equation (15), and we collect the values of the exponents in table 1. It is satisfying to see that in 1D we find a good agreement with the results of a Monte Carlo study [32] on the same model[9]. While we find evidence for the exponent equality $\lambda_\Gamma = \lambda_R$ in close analogy to magnetic systems quenched to criticality from a fully disordered state, the ageing exponents a and b are different and we conjecture

$$1 + a = b = 2\delta \tag{21}$$

where δ is a well-known non-equilibrium exponent. In particular, this implies that the quasi-stationary regime of the contact process is already out of equilibrium and the recent attempt [25] to define a genuine non-equilibrium temperature does not go through. Instead of the

[8] Recently, it has been established that in those ageing systems which undergo cluster dilution rather than domain growth, there is a universal early-time regime where dynamical scaling does *not* hold [51]. On the other hand, from Monte Carlo simulations we know that in the contact process ageing proceeds via cluster dissolution [32, figure 2]. Therefore, a small deviation of the numerical data from the t/s scaling for $t/s \approx 1$ is to be expected.

[9] This agreement is not completely trivial, since the initial densities in these two studies are different.

fluctuation–dissipation theorem valid for equilibrium systems, we have found evidence that in the contact process rather the ratio $R(t,s)/\Gamma(t,s)$ should converge to a finite value in the limit of widely separated times (see figure 15).

On the other hand, we have seen that the scaling form of the response function is in agreement with local scale invariance. We point out that this confirmation is obtained in a formulation based on the master equation and not in a field-theoretical setting based on a Langevin equation. All existing confirmations of local scale invariance, see [7], have either been obtained in this set-up or else come from models which reduce to free field theories.

Acknowledgments

We thank A Gambassi, M Pleimling, J Ramasco, M A Santos and C A da Silva Santos for useful discussions and correspondence.

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